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Almost Kähler Walker 4-manifolds

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Abstract

It is shown that any proper almost Hermitian structure on a Walker 4-manifold is isotropic Kähler. Moreover, a local description of proper almost Kähler structures that are self-dual, $*$-Einstein or Einstein is given and it is proved that any proper strictly almost Kähler Einstein structure is self-dual, Ricci flat and $*$-Ricci flat. This is used to supply examples of flat indefinite non-Kähler almost Kähler structures.

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1. Introduction

An almost Hermitian structure on a manifold M consists of a nondegenerate 2-form Ω, an almost complex structure J and a metric g satisfying the compatibility condition $\Omega(X, Y) = g(JX, Y)$. If the 2-form Ω is closed (i.e., it is a symplectic form) the structure is said to be almost Kähler and (g, J) is said to be Kähler if, in addition, the almost complex structure J is integrable (i.e., it is defined by a complex coordinate atlas on M). It is worth emphasizing that each two of the objects (g, J, Ω) determine the third one. However, whenever the starting point is a symplectic structure Ω, there are many different pairs (g, J) of almost Hermitian structures sharing the same Kähler form Ω.

A long standing problem in almost Hermitian geometry is that of relating the properties of the structure (g, J, Ω) to the curvature of (M, g). For example the Goldberg conjecture [14], which claims that compact almost Kähler Einstein manifolds are necessarily Kähler, is still an open problem. (See the survey [1] for an update on the integrability of almost Kähler structures.) Although the Goldberg conjecture is of global nature it is already known that some additional curvature conditions suffice to show the integrability of the almost complex structure at the local level.

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For instance, in dimension 4, Einstein almost Kähler metrics which are $*$-Einstein are necessarily Kähler [24]. (The $*$-Einstein condition can be replaced by the second Gray curvature identity or the anti-self-duality condition and the integrability still follows [1].) It is also well known [2,24,25] that any almost Kähler metric of constant sectional curvature is Kähler (and flat). Attention should be paid to the fact that all the above results are true in the Riemannian setting (i.e., the induced metric $g(\cdot,\cdot) = \Omega(J\cdot,\cdot)$ is positive definite). Proofs usually make use of relations involving some curvature terms (e.g. $\tau - \tau^* = \frac{1}{2}\|\nabla\Omega\|^2$) from where one obtains $\|\nabla\Omega\|^2 = 0$, which shows the desired integrability (see, for example, [27]). Even though such identities remain valid in the indefinite setting, one may expect the condition $\|\nabla\Omega\|^2 = 0$ to define a class of indefinite almost Hermitian structures strictly containing the Kähler ones, and this is indeed the case (see [4,12]). Our purpose in this work is to show that the class of isotropic Kähler structures is larger than expected and it provides examples showing that the results mentioned above are not true for indefinite metrics. To do this we consider Walker metrics [28] on 4-manifolds together with the so-called proper almost complex structures [22] and obtain a local description of those metrics which are almost Kähler and self-dual, $*$-Einstein or Einstein. As a consequence we show that any proper strictly almost Kähler Einstein structure is self-dual, Ricci flat and $*$-Ricci flat. Note that an indefinite strictly almost Kähler Einstein metric on an eight-dimensional torus has been recently reported in [23].

As a notational fact, since throughout this paper we only deal with metrics of signature $(++--)$, the word indefinite will be omitted in what follows.

2. Preliminaries

Throughout this paper we use the following convention for the curvature tensor $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y]$, where ∇ denotes the Levi-Civita connection. $\rho(X,Y) = \operatorname{trace}\{U \rightsquigarrow R(X,U)Y\}$ and $\tau = \operatorname{trace}\rho$ are the Ricci tensor and the scalar curvature, respectively. As usual, (M,g) is said to be Einstein if $\rho = \frac{\tau}{n}g$, $n = \dim M$, in which case the scalar curvature is necessarily constant. A special class of Einstein manifolds is that of Osserman ones, i.e., those pseudo-Riemannian manifolds whose Jacobi operators $R_X = R(X,\cdot)X$ have eigenvalues independent of the direction and the basepoint. (See, for example, [5,10,11,13] and the references therein for more information.) Osserman metrics have a special significance in dimension 4, since an algebraic curvature tensor on a four-dimensional vector space is Osserman if and only if it is Einstein and self-dual (or anti-self-dual) [11,13] (see also [6]).

Associated with an almost Hermitian structure (g,J) we consider the $*$-Ricci tensor defined by $\rho^*(X,Y) = \operatorname{trace}\{U \rightsquigarrow -\frac{1}{2}JR(X,JY)U\}$ and the $*$-scalar curvature $\tau^* = \operatorname{trace}\rho^*$. Note that ρ and ρ^* coincide in the Kähler setting but ρ^* is not symmetric in general. An n-dimensional almost Hermitian manifold (M,g,J) is called *weakly $*$-Einstein* if $\rho^* = \frac{\tau^*}{n}g$ and is said to be *$*$-Einstein* if, in addition, τ^* is constant.

2.1. Walker metrics

A *Walker manifold* is a triple $(M,g,\mathcal{D})$ where M is an n-dimensional manifold, g an indefinite metric and $\mathcal{D}$ an r-dimensional parallel null distribution. Of special interest are those manifolds admitting a field of null planes of maximum dimension $r = \frac{n}{2}$. Since the dimension of a null plane is $r \leq \frac{n}{2}$, the lowest possible case is that of $(++--)$-manifolds admitting a field of parallel null 2-planes.

For our purposes it is convenient to use special coordinate systems associated with any Walker metric. Recall that, by a result of Walker [28], for every Walker metric g on a 4-manifold M, there exist local coordinates (x,y,z,t) around any point of M such that the matrix of g in these coordinates has the following form:

$$g_{(x,y,z,t)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix} \tag{1}$$

for some functions a, b and c depending on the coordinates (x,y,z,t). As a matter of notation, throughout this work we denote by ∂_i the coordinate tangent vectors, $i = x,\ldots,t$. Also, h_i means partial derivative $\frac{\partial h}{\partial i}$, $i = x,\ldots,t$, for any function $h(x,y,z,t)$.

Observe that Walker metrics appear as the underlying structure of several specific pseudo-Riemannian structures. Some of those, as in the examples below, clearly motivate the investigation of pseudo-Riemannian manifolds carrying a parallel degenerate plane field. Moreover, indecomposable metrics of neutral signature which are not irreducible play a distinguished role in investigating the holonomy of indefinite metrics. Those metrics are naturally equipped with a Walker structure (see for example [3] and the references therein).

2.1.1. Two-step nilpotent Lie groups with degenerate center

Let N be a two-step nilpotent Lie group with left-invariant pseudo-Riemannian metric tensor $\langle\cdot,\cdot\rangle$ and Lie algebra $\mathfrak{n}$. In the Riemannian case, one splits $\mathfrak{n}=\mathfrak{z}\oplus\mathfrak{z}^{\perp}$ where the superscript denotes the orthogonal complement with respect to the inner product and $\mathfrak{z}$ stands for the center of $\mathfrak{n}$. In the pseudo-Riemannian case, however, $\mathfrak{z}$ may contain a degenerate subspace $\mathfrak{U}$ for which $\mathfrak{U}\subseteq\mathfrak{U}^{\perp}$. Hence the following decomposition is introduced in [7]:

$$\mathfrak{n}=\mathfrak{z}\oplus\mathfrak{v}=\mathfrak{U}\oplus\mathfrak{Z}\oplus\mathfrak{D}\oplus\mathfrak{E}$$

in which $\mathfrak{z}=\mathfrak{U}\oplus\mathfrak{Z}$ and $\mathfrak{v}=\mathfrak{D}\oplus\mathfrak{E}$. Here $\mathfrak{U}$ and $\mathfrak{D}$ are complementary null subspaces and $\mathfrak{U}^{\perp}\cap\mathfrak{D}^{\perp}=\mathfrak{Z}\oplus\mathfrak{E}$. (Indeed, $\mathfrak{Z}$ is the part of the center in $\mathfrak{U}^{\perp}\cap\mathfrak{D}^{\perp}$ and $\mathfrak{E}$ is its orthogonal complement in $\mathfrak{U}^{\perp}\cap\mathfrak{D}^{\perp}$.) The geometry of a pseudo-Riemannian two-step nilpotent Lie group is essentially controlled by the linear mapping $j:\mathfrak{U}\oplus\mathfrak{Z}\to\mathrm{End}(\mathfrak{D}\oplus\mathfrak{E})$ defined by $\langle j(a)x,y\rangle=\langle[x,y],\imath a\rangle$, where $\imath$ is an involution interchanging $\mathfrak{U}$ and $\mathfrak{D}$. Now, since $[\mathfrak{n},\mathfrak{n}]\subseteq\mathfrak{z}$, it immediately follows that $\mathfrak{U}$ is a parallel degenerate subspace and thus the metric $\langle\cdot,\cdot\rangle$ is necessarily a Walker one.

On the other hand, note that four-dimensional indefinite Kähler Lie algebras $\mathfrak{g}$ naturally split into two classes depending on whether a naturally defined Lagrangian ideal $\mathfrak{h}$ satisfying $\mathfrak{h}\cap J\mathfrak{h}$ is trivial or $\mathfrak{h}\cap J\mathfrak{h}$ coincides with $\mathfrak{g}$. If the second possibility occurs, then the induced metric is a Walker one. Such Lie algebras correspond to the cases $\mathbb{R}\times\mathfrak{h}_3$, $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{r}_{4,-1,-1}$, $\delta_{4,1}$ and $\delta_{4,2}$. (See [26] for details.)

2.1.2. Para-Kähler and hyper-symplectic structures

A para-Kähler manifold is a symplectic manifold admitting two transversal Lagrangian foliations (see [8,17]). Such a structure induces a decomposition of the tangent bundle TM into the Whitney sum of Lagrangian subbundles L and L', that is, $TM=L\oplus L'$. By generalizing this definition, an almost para-Hermitian manifold is defined to be an almost symplectic manifold (M,Ω) whose tangent bundle splits into the Whitney sum of Lagrangian subbundles. This definition implies that the (1, 1)-tensor field J defined by $J=\pi_L-\pi_{L'}$ is an almost paracomplex structure, that is $J^2=id$ on M, such that $\Omega(JX,JY)=-\Omega(X,Y)$ for all $X,Y\in\Gamma TM$, where π_L and π'_L are the projections of TM onto L and L', respectively. The 2-form Ω induces a nondegenerate (0, 2)-tensor field g on M defined by $g(X,Y)=\Omega(X,JY)$, where $X,Y\in\Gamma TM$. Now the relation between the almost paracomplex and the almost symplectic structures on M shows that g defines a pseudo-Riemannian metric of signature (n,n) on M and, moreover, $g(JX,Y)+g(X,JY)=0$, where $X,Y\in\Gamma TM$. The special significance of the para-Kähler condition is equivalently stated in terms of the parallelizability of the paracomplex structure with respect to the Levi-Civita connection of g, that is $\nabla J=0$. The paracomplex structure J has eigenvalues ±1 with completely degenerate corresponding eigenspaces due to the skew-symmetry of J. Moreover, since J is parallel in the para-Kähler setting, so are the ±1-eigenspaces, which shows that any para-Kähler structure (g,J) has necessarily an underlying Walker metric.

An almost hyper-paracomplex structure on a $4n$-dimensional manifold M is a triple J_a, $a=1,2,3$, where J_1, J_2 are almost paracomplex structures and J_3 is an almost complex structure, satisfying the paraquaternionic identities

$$J_1^2=J_2^2=-J_3^2=1,\qquad J_1J_2=-J_2J_1=J_3.$$

We note that on an almost hyper-paracomplex manifold there is actually a two-sheeted hyperboloid of almost complex structures: $S_1^2(-1)=\{c_1J_1+c_2J_2+c_3J_3:c_1^2+c_2^2-c_3^2=-1\}$ and a one-sheeted hyperboloid of almost paracomplex structures: $S_1^2(1)=\{b_1J_1+b_2J_2+b_3J_3:b_1^2+b_2^2-b_3^2=1\}$. A hyper-para-Hermitian metric is a pseudo-Riemannian metric which is compatible with the (almost) hyper-paracomplex structure in the sense that the metric g is skew-symmetric with respect to each J_a, $a=1,2,3$, i.e.

$$g(J_1.,J_1.)=g(J_2.,J_2.)=-g(J_3.,J_3.)=-g(.,.).$$

Such a structure is called an (almost) hyper-para-Hermitian structure. If on a hyper-para-Hermitian manifold there exists an admissible basis such that each J_a, $a=1,2,3$, is parallel with respect to the Levi-Civita connection or, equivalently, the three Kähler forms are closed, then the manifold is called hyper-symplectic [16]. In this case J_1 and J_2 are para-Kähler structures and it follows that g is a Walker metric.

2.1.3. Hypersurfaces with nilpotent shape operators

Einstein hypersurfaces M in indefinite real space forms $\overline{M}(c)$ have been studied by Magid [19], who showed that the shape operator S of any such hypersurface is diagonalizable; it defines, after rescaling, a complex structure on M (i.e., $S^2 = -b^2 Id$ for some $b \neq 0$), or it is two-step nilpotent (i.e., $S^2 = 0$, $S \neq 0$). Since S is a self-adjoint operator, its kernel is a completely degenerate subspace. Moreover the fact that S is parallel shows that the underlying metric on M is a Walker one [20].

2.1.4. Four-dimensional Osserman metrics

Finally note that Walker metrics also appear associated with some curvature problems. Let (M, g) be a pseudo-Riemannian metric of signature $(++--)$. Then, for each non-null vector X, the induced metric on $X^\perp$ is of Lorentzian signature and thus the Jacobi operator $R_X = R(X, \cdot)X$, viewed as an endomorphism of $X^\perp$, corresponds to one of the following possibilities [5] (see also [18]):

$$\underset{\textit{Type Ia}}{\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}}, \quad \underset{\textit{Type Ib}}{\begin{pmatrix} \alpha & -\beta & \\ \beta & \alpha & \\ & & \gamma \end{pmatrix}}, \quad \underset{\textit{Type II}}{\begin{pmatrix} \alpha & & \\ & \beta & \\ & 1 & \beta \end{pmatrix}}, \quad \underset{\textit{Type III}}{\begin{pmatrix} \alpha & & \\ 1 & \alpha & \\ & 1 & \alpha \end{pmatrix}}.$$

Type Ia Osserman metrics correspond to real, complex and paracomplex space forms, Type Ib Osserman metrics do not exist [5] and Type II Osserman metrics with non-nilpotent Jacobi operators have recently been classified [10]. Further, note that any Type II Osserman metric whose Jacobi operators have nonzero eigenvalues is necessarily a Walker metric.

3. Proper almost complex structure

It is well known that the existence of a metric of signature $(++--)$ with structure group $SO_0(2, 2)$ is equivalent to the existence of a pair of commuting almost complex structures [21], and, moreover, any such pseudo-Riemannian metric may be viewed as an indefinite almost Hermitian metric for a suitable almost complex structure. Such almost complex structures are not uniquely determined. One such structure associated with any four-dimensional Walker metric has been locally given in [22] and called the proper almost complex structure. Our purpose here is to investigate curvature properties of Walker metrics by considering the associated proper structures. It turns out that these structures exhibit a very rich behavior and provide examples of non-Kähler self-dual Einstein almost Kähler and Hermitian structures. It is important to recognize that such exceptional behavior comes from the fact that any proper almost Hermitian structure is isotropic Kähler but not necessarily Kähler.

Next, for a Walker metric (1) an orthonormal basis can be specialized by using the canonical coordinates as follows:

$$\begin{aligned} e_1 &= \frac{1}{2}(1-a)\partial_x + \partial_z, \qquad e_2 = -c\partial_x + \frac{1}{2}(1-b)\partial_y + \partial_t, \\ e_3 &= -\frac{1}{2}(1+a)\partial_x + \partial_z, \qquad e_4 = -c\partial_x - \frac{1}{2}(1+b)\partial_y + \partial_t. \end{aligned} \tag{2}$$

With respect to the local frame above, the metric is diagonal $[1, 1, -1, -1]$, and hence a natural almost complex structure J can be defined setting

$$J = e_2 \otimes e^1 - e_1 \otimes e^2 + e_4 \otimes e^3 - e_3 \otimes e^4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This structure on a Walker 4-manifold, called *proper* in [22], induces a positive $\frac{\pi}{2}$-rotation on the degenerate parallel field $\mathcal{D}$ spanned by ∂_x, ∂_y. The proper almost complex structure is completely determined by the metric as follows [22]:

$$\begin{aligned} J\partial_x &= \partial_y, \qquad J\partial_z = -c\partial_x + \frac{1}{2}(a-b)\partial_y + \partial_t, \\ J\partial_y &= -\partial_x, \qquad J\partial_t = \frac{1}{2}(a-b)\partial_x + c\partial_y - \partial_z. \end{aligned} \tag{3}$$

The space of linear invariants of an almost Hermitian manifold (M^n, g, J) is given by $I_n = \{\|\nabla\Omega\|^2, \|d\Omega\|^2, \|\delta\Omega\|^2, \|N_J\|^2, \tau, \tau^*\}$, where $N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$ denotes the Nijenhuis tensor of the almost complex structure J, with

$$\|\nabla\Omega\|^2 = \sum_{a,b,c=1}^{n} \varepsilon_a\varepsilon_b\varepsilon_c (\nabla_{e_a}\Omega)(e_b, e_c)^2, \qquad \|d\Omega\|^2 = \sum_{a,b,c=1}^{n} \varepsilon_a\varepsilon_b\varepsilon_c d\Omega(e_a, e_b, e_c)^2,$$

$$\|\delta\Omega\|^2 = \sum_{a=1}^{n} \varepsilon_a \delta\Omega(e_a)^2, \qquad \|N_J\|^2 = \sum_{a,b,c=1}^{n} \varepsilon_a\varepsilon_b \|N_J(e_a, e_b)\|^2,$$

$$\tau = \sum_{a,b=1}^{n} \varepsilon_a\varepsilon_b R(e_a, e_b, e_a, e_b), \qquad \tau^* = \frac{1}{2}\sum_{a,b=1}^{n} \varepsilon_a\varepsilon_b R(e_a, Je_a, e_b, Je_b),$$

and $\{e_1, \ldots, e_n\}$ is a local orthonormal basis [15]. Further, note that if (M, g, J) is four-dimensional, then $I_4 = \{\|\nabla\Omega\|^2, \|N_J\|^2, \tau, \tau^*\}$.

Recall that an indefinite almost Hermitian structure (g, J) is said to be *isotropic Kähler* if $\|\nabla J\|^2 = 0$ but $\nabla J \neq 0$. Examples of isotropic Kähler structures were given first in [12] in dimension 4 and subsequently in [4] in dimension 6. Our purpose in this section is to show that the class of isotropic Kähler structures is larger than expected. For instance, any proper almost Hermitian structure on a Walker 4-manifold is isotropic Kähler as we will see in Theorem 1.

Theorem 1. *Any proper almost Hermitian structure* (g, J) *on a Walker 4-manifold satisfies* $\|\nabla\Omega\|^2 = 0$, $\|d\Omega\|^2 = 0$, $\|\delta\Omega\|^2 = 0$ *and* $\|N_J\|^2 = 0$.

Moreover the scalar and $*$*-scalar curvatures are given by* $\tau = a_{xx} + b_{yy} + 2c_{xy}$ *and* $\tau^* = -a_{yy} - b_{xx} + 2c_{xy}$, *respectively.*

Proof. For the Nijenhuis tensor N_J associated with J, put $N_{ij} = N_J(\partial_i, \partial_j)$. Then, after some calculations one has from (3) that

$$N_{xz} = -N_{yt} = \frac{1}{2}(a_x - b_x - 2c_y)\partial_x + \frac{1}{2}(a_y - b_y + 2c_x)\partial_y,$$
$$N_{xt} = N_{yz} = \frac{1}{2}(a_y - b_y + 2c_x)\partial_x - \frac{1}{2}(a_x - b_x - 2c_y)\partial_y,$$
$$\begin{aligned} N_{zt} \ &= \ \frac{1}{4}((a-b)(a_y - b_y + 2c_x) - 2c(a_x - b_x - 2c_y))\partial_x \\ &\quad - \frac{1}{4}((a-b)(a_x - b_x - 2c_y) + 2c(a_y - b_y + 2c_x))\partial_y. \end{aligned}$$

Now, a straightforward calculation using the fact that the inverse of the metric tensor, $g^{-1} = (g^{\alpha\beta})$, is given by

$$g^{-1} = \begin{pmatrix} -a & -c & 1 & 0 \\ -c & -b & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

shows that $\|N_J\|^2 = \sum_{ijkl} g^{ij} g^{kl} g(N_{ik}, N_{jl}) = 0$.

The Levi-Civita connection of a Walker metric (1) is given by (see, for example, [10])

$$\nabla_{\partial_x}\partial_z = \frac{1}{2}a_x\partial_x + \frac{1}{2}c_x\partial_y, \qquad \nabla_{\partial_x}\partial_t = \frac{1}{2}c_x\partial_x + \frac{1}{2}b_x\partial_y,$$
$$\nabla_{\partial_y}\partial_z = \frac{1}{2}a_y\partial_x + \frac{1}{2}c_y\partial_y, \qquad \nabla_{\partial_y}\partial_t = \frac{1}{2}c_y\partial_x + \frac{1}{2}b_y\partial_y,$$
$$\nabla_{\partial_z}\partial_z = \frac{1}{2}(aa_x + ca_y + a_z)\partial_x + \frac{1}{2}(ca_x + ba_y - a_t + 2c_z)\partial_y - \frac{a_x}{2}\partial_z - \frac{a_y}{2}\partial_t,$$
$$\nabla_{\partial_z}\partial_t = \frac{1}{2}(a_t + ac_x + cc_y)\partial_x + \frac{1}{2}(b_z + cc_x + bc_y)\partial_y - \frac{c_x}{2}\partial_z - \frac{c_y}{2}\partial_t,$$
$$\nabla_{\partial_t}\partial_t = \frac{1}{2}(ab_x + cb_y - b_z + 2c_t)\partial_x + \frac{1}{2}(cb_x + bb_y + b_t)\partial_y - \frac{b_x}{2}\partial_z - \frac{b_y}{2}\partial_t.$$

For the covariant derivative ∇J of the almost complex structure put $(\nabla J)_{ij} = (\nabla_{\partial_i} J)\partial_j$. Then, after some calculations we obtain

$$
\begin{aligned}
(\nabla J)_{zx} &= \frac{1}{2}(a_y + c_x)\partial_x - \frac{1}{2}(a_x - c_y)\partial_y,\\
(\nabla J)_{zy} &= \frac{1}{2}(c_y - a_x)\partial_x - \frac{1}{2}(a_y + c_x)\partial_y,\\
(\nabla J)_{zz} &= \frac{1}{2}(a(a_y + c_x) - c(a_x - c_y))\partial_x - \frac{1}{4}(a+b)(a_x - c_y)\partial_y - \frac{1}{2}(a_y + c_x)\partial_z + \frac{1}{2}(a_x - c_y)\partial_t,\\
(\nabla J)_{zt} &= \frac{1}{4}(a+b)(c_y - a_x)\partial_x - \frac{1}{2}(b(a_y + c_x) + c(a_x - c_y))\partial_y + \frac{1}{2}(a_x - c_y)\partial_z + \frac{1}{2}(a_y + c_x)\partial_t,\\
(\nabla J)_{tx} &= \frac{1}{2}(b_x + c_y)\partial_x + \frac{1}{2}(b_y - c_x)\partial_y,\\
(\nabla J)_{ty} &= \frac{1}{2}(b_y - c_x)\partial_x - \frac{1}{2}(b_x + c_y)\partial_y,\\
(\nabla J)_{tz} &= \frac{1}{2}(a(b_x + c_y) + c(b_y - c_x))\partial_x + \frac{1}{4}(a+b)(b_y - c_x)\partial_y - \frac{1}{2}(b_x + c_y)\partial_z - \frac{1}{2}(b_y - c_x)\partial_t,\\
(\nabla J)_{tt} &= \frac{1}{4}(a+b)(b_y - c_x)\partial_x - \frac{1}{2}(b(b_x + c_y) - c(b_y - c_x))\partial_y - \frac{1}{2}(b_y - c_x)\partial_z + \frac{1}{2}(b_x + c_y)\partial_t.
\end{aligned}
$$

Now a long but straightforward calculation shows that

$$\|\nabla J\|^2 = \sum_{i,j,k,l} g^{ij} g^{kl} g((\nabla J)_{ik}, (\nabla J)_{jl}) = 0.$$

Then, it follows that $\|\nabla \Omega\|^2 = \|d\Omega\|^2 = \|N_J\|^2 = 0$ since for an arbitrary almost Hermitian 4-manifold, one has the identities

$$\|\delta\Omega\|^2 = \frac{1}{6}\|d\Omega\|^2, \qquad \|\nabla\Omega\|^2 = \frac{1}{3}\|d\Omega\|^2 + \frac{1}{4}\|N_J\|^2.$$

Finally, the expressions of the scalar and $*$-scalar curvatures can be obtained by means of the formulas for the curvature tensor of a Walker metric given, for example, in [10,22]. □

Remark 1. Examples of compact isotropic Kähler structures can be constructed on tori taking a, b and c in (1) to be periodic functions on $\mathbb{R}^4$. Moreover note that in the general situation the isotropic Kähler structures (1) and (3) are neither complex nor symplectic. Indeed, according to [22], the proper almost Hermitian structure (g, J) is:

- almost Kähler if and only if

$$a_x + b_x = 0, \qquad a_y + b_y = 0, \tag{4}$$

- Hermitian if and only if

$$a_x - b_x = 2c_y, \qquad a_y - b_y = -2c_x, \tag{5}$$

- Kähler if and only if

$$a_x = -b_x = c_y, \qquad a_y = -b_y = -c_x. \tag{6}$$

Hence, for special choices of functions (which may still be assumed to be periodic) satisfying (4) or (5), examples of symplectic or integrable isotropic Kähler structures can be given.

4. Almost Kähler self-dual proper structures

Considering the Riemann curvature tensor as an endomorphism of $\Lambda^2(M)$, we have the following $O(2,2)$-decomposition:

$$R \equiv \frac{\tau}{12} Id_{\Lambda^2} + \rho_0 + W : \Lambda^2 \to \Lambda^2, \tag{7}$$

where W denotes the Weyl conformal curvature tensor and ρ_0 the traceless Ricci tensor, $\rho_0(X,Y) = \rho(X,Y) - \frac{\tau}{4}\,g(X,Y)$. The Hodge star operator $\star : \Lambda^2 \to \Lambda^2$ associated with any $(++--)$-metric induces a further splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, where $\Lambda^2_\pm$ denotes the ± 1-eigenspaces of the Hodge star operator, that is $\Lambda^2_\pm = \{\alpha \in \Lambda^2(M) / \star\alpha = \pm\alpha\}$. Correspondingly, the curvature tensor decomposes as $R \equiv \frac{\tau}{12}\,Id_{\Lambda^2} + \rho_0 + W^+ + W^-$, where $W^\pm = \frac{1}{2}\,(W \pm \star W)$. Recall that a pseudo-Riemannian 4-manifold is called *self-dual* (resp., *anti-self-dual*) if $W^- = 0$ (resp., $W^+ = 0$).

Let $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis given by (2). Then $\Lambda^2_\pm = \langle\{E_1^\pm, E_2^\pm, E_3^\pm\}\rangle$, where

$$E_1^\pm = \frac{e^1 \wedge e^2 \pm e^3 \wedge e^4}{\sqrt{2}}, \qquad E_2^\pm = \frac{e^1 \wedge e^3 \pm e^2 \wedge e^4}{\sqrt{2}}, \qquad E_3^\pm = \frac{e^1 \wedge e^4 \mp e^2 \wedge e^3}{\sqrt{2}}.$$

Here observe that $e^i \wedge e^j \wedge \star(e^k \wedge e^l) = (\delta^i_k\delta^j_l - \delta^i_l\delta^j_k)\varepsilon_i\varepsilon_j e^1 \wedge e^2 \wedge e^3 \wedge e^4$, where $\varepsilon_i = g(e_i, e_i)$. Further, note that $\langle E_1^\pm, E_1^\pm\rangle = 1$, $\langle E_2^\pm, E_2^\pm\rangle = -1$, $\langle E_3^\pm, E_3^\pm\rangle = -1$, and therefore the self-dual and anti-self-dual Weyl curvature operators $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$ have the following matrix form with respect to the bases above:

$$W^\pm = \begin{pmatrix} W_{11}^\pm & W_{12}^\pm & W_{13}^\pm \\ -W_{12}^\pm & -W_{22}^\pm & -W_{23}^\pm \\ -W_{13}^\pm & -W_{23}^\pm & -W_{33}^\pm \end{pmatrix}, \tag{8}$$

where $W_{ij}^\pm = W(E_i^\pm, E_j^\pm)$ and $W(e^i \wedge e^j, e^k \wedge e^l) = W(e_i, e_j, e_k, e_l)$.

Self-dual Walker metrics have been previously investigated in [10] (see also [9]) showing that a metric (1) is self-dual if and only if the functions a, b, c have the forms

$$\begin{aligned} a(x,y,z,t) &= x^3\mathcal{A} + x^2\mathcal{B} + x^2y\mathcal{C} + xy\mathcal{D} + xP + yQ + \xi, \\ b(x,y,z,t) &= y^3\mathcal{C} + y^2\mathcal{E} + xy^2\mathcal{A} + xy\mathcal{F} + xS + yT + \eta, \\ c(x,y,z,t) &= \frac{1}{2}x^2\mathcal{F} + \frac{1}{2}y^2\mathcal{D} + x^2y\mathcal{A} + xy^2\mathcal{C} + \frac{1}{2}xy(\mathcal{B} + \mathcal{E}) + xU + yV + \gamma, \end{aligned} \tag{9}$$

where all capital, calligraphic and Greek letters stand for arbitrary smooth functions depending only on the coordinates (z, t).

Anti-self-dual Walker metrics are much more difficult to describe since the self-dual part of the Weyl curvature operator is given by

$$W^+ = \begin{pmatrix} W_{11}^+ & W_{12}^+ & W_{11}^+ + \frac{\tau}{12} \\ -W_{12}^+ & \frac{\tau}{6} & -W_{12}^+ \\ -W_{11}^+ - \frac{\tau}{12} & -W_{12}^+ & -W_{11}^+ - \frac{\tau}{6} \end{pmatrix},$$

where the expressions of W_{11}^+ and W_{12}^+ are as follows:

$$\begin{aligned} W_{11}^+ = \; & \frac{1}{12}(6ca_xb_y - 6a_xb_z - 6ba_xc_y + 12a_xc_t - 6ca_yb_x + 6a_yb_t + 6ba_yc_x \\ & + 6a_zb_x - 6a_tb_y - 12a_tc_x + 6ab_xc_y - 6ab_yc_x + 12b_yc_z - 12b_zc_y \\ & - a_{xx} - 12c^2a_{xx} - 12bca_{xy} + 24ca_{xt} - 3b^2a_{yy} + 12ba_{yt} - 12a_{tt} \\ & - 3a^2b_{xx} + 12ab_{xz} - b_{yy} - 12b_{zz} + 12acc_{xx} - 2c_{xy} + 6abc_{xy} \\ & - 24cc_{xz} - 12ac_{xt} - 12bc_{yz} + 24c_{zt}), \\ W_{12}^+ = \; & \frac{1}{4}(ac_{xx} + ab_{xy} - ba_{xy} - bc_{yy} + 2(a_{xt} - b_{yz} - c_{xz} + c_{yt} - ca_{xx} - cc_{xy})). \end{aligned} \tag{10}$$

Hence W^+ has eigenvalues $\left\{\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}\right\}$ and, moreover, it is diagonalizable if and only if $\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2 = 0$ (cf. [10]).

Observe that the complex structure induces the orientation opposite to that defined by the Kähler form. (Indeed, the Kähler form associated with the proper almost Hermitian structure corresponds to the bivector $\sqrt{2}E_1^-$.)

Theorem 2. *A proper almost Kähler structure (g, J) on a Walker 4-manifold is self-dual if and only if*

$$\begin{aligned} a &= xy\mathcal{D} + xP + yQ + \xi, \\ b &= -xy\mathcal{D} - xP - yQ + \eta, \\ c &= -\frac{1}{2}x^2\mathcal{D} + \frac{1}{2}y^2\mathcal{D} + xU + yV + \gamma. \end{aligned} \tag{11}$$

Proof. It is immediate from (4) and (9). □

5. Almost Kähler ∗-Einstein proper structures

Given a proper almost Hermitian structure (g, J), defined by (1) and (3), one can compute its ∗-Ricci tensor making use of the curvature formulas in [10,22]. The ∗-Einstein equation ($\rho_0^* = \rho^* - \frac{\tau^*}{4}g = 0$) for a proper almost Hermitian structure can be written as a system of PDEs as follows:

$$\begin{aligned} &(\rho_0^*)_{xz} = -(\rho_0^*)_{yt} = -(\rho_0^*)_{zx} = (\rho_0^*)_{ty} = \frac{1}{4}(a_{yy} - b_{xx}) = 0, \\ &(\rho_0^*)_{xt} = -(\rho_0^*)_{zy} = -\frac{1}{2}(a_{xy} - c_{xx}) = 0, \\ &(\rho_0^*)_{yz} = -(\rho_0^*)_{tx} = -\frac{1}{2}(b_{xy} - c_{yy}) = 0, \\ &(\rho_0^*)_{zz} = \frac{1}{4}\Big\{a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2 + 2c(a_{xy} - c_{xx}) \\ &\qquad + ba_{yy} - 2a_{yt} + ab_{xx} - 2b_{xz} - (a+b)c_{xy} + 2c_{xt} + 2c_{yz}\Big\} = 0, \\ &(\rho_0^*)_{zt} = -\frac{1}{4}\left\{(a-b)(a_{xy} - c_{xx}) + c(a_{yy} - b_{xx})\right\} = 0, \\ &(\rho_0^*)_{tz} = \frac{1}{4}\left\{(a-b)(b_{xy} - c_{yy}) + c(a_{yy} - b_{xx})\right\} = 0, \\ &(\rho_0^*)_{tt} = \frac{1}{4}\Big\{a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2 + 2c(b_{xy} - c_{yy}) \\ &\qquad + ba_{yy} - 2a_{yt} + ab_{xx} - 2b_{xz} - (a+b)c_{xy} + 2c_{xt} + 2c_{yz}\Big\} = 0. \end{aligned} \tag{12}$$

Note that the ∗-scalar curvature is given by

$$\tau^* = -a_{yy} - b_{xx} + 2\,c_{xy}. \tag{13}$$

Theorem 3. *A proper almost Hermitian structure (g, J) is almost Kähler and ∗-Einstein if and only if the functions a, b and c have the forms*

$$\begin{aligned} a &= (x^2 - y^2)\kappa + xP(z,t) + yQ(z,t) + \xi(z,t), \\ b &= (y^2 - x^2)\kappa - xP(z,t) - yQ(z,t) + \eta(z,t), \\ c &= 2xy\kappa + xU(z,t) + yV(z,t) + \gamma(z,t), \end{aligned} \tag{14}$$

where κ is a constant and

$$2(P_z + V_z - Q_t + U_t) = (P - V)^2 + (Q + U)^2 + 4\kappa(\xi + \eta). \tag{15}$$

In this case the scalar and ∗-scalar curvatures are constant, $\tau = \tau^* = 8\kappa$.

Proof. It follows from (4) and (12) that (g, J) is almost Kähler and ∗-Einstein if and only if

$$a_x + b_x = a_y + b_y = 0, \qquad a_{xy} = c_{xx}, \qquad b_{xy} = c_{yy}, \qquad a_{yy} = b_{xx}, \tag{16}$$

and

$$a_x b_x + a_y b_y - a_y c_x + b_y c_x + a_x c_y - b_x c_y - c_x^2 - c_y^2 + b a_{yy} - 2a_{yt} + a b_{xx} - 2b_{xz} - (a+b)c_{xy} + 2c_{xt} + 2c_{yz} = 0. \quad (17)$$

Using (16) we easily get that

$$a + b = T(z,t), \qquad c_{xx} = a_{xy} = -c_{yy}, \qquad a_{xx} + a_{yy} = 0,$$

where T is a smooth function depending only on z and t. Hence $c_x = a_y + A(y,z,t)$, $c_y = -a_x + B(x,z,t)$ for some smooth functions A, B, and the equation $a_{xx} + a_{yy} = 0$ implies that $A_y = B_x$. Thus

$$A(y,z,t) = y\alpha(z,t) + \beta(z,t), \qquad B(x,z,t) = x\alpha(z,t) + \delta(z,t),$$

where α, β, δ are smooth functions. Therefore the system (16) is equivalent to

$$c_x = a_y + y\alpha(z,t) + \beta(z,t), \qquad c_y = -a_x + x\alpha(z,t) + \delta(z,t),$$
$$a_{xx} + a_{yy} = 0, \qquad b = T(z,t) - a. \quad (18)$$

Now a straightforward computation making use of (18) shows that the condition (17) can be written as

$$(2a_y + y\alpha + \beta)^2 + (2a_x - x\alpha - \delta)^2 = 2x\alpha_z + 2y\alpha_t + 2\delta_z + 2\beta_t - \alpha T. \quad (19)$$

Set

$$H(x,y,z,t) = 2a(x,y,z,t) - \frac{1}{2}(x^2 - y^2)\alpha(z,t) - x\delta(z,t) + y\beta(z,t). \quad (20)$$

Then (19) takes the form

$$H_x^2 + H_y^2 = 2x\alpha_z + 2y\alpha_t + 2\delta_z + 2\beta_t - \alpha T. \quad (21)$$

The latter identity shows that, for any fixed z and t, the right-hand side of (21) is a non-negative linear function of x and y. This implies that the coefficients α_z and α_t vanish, i.e. the function $\alpha(z,t)$ is constant. Moreover, using the fact that $a_{xx} + a_{yy} = 0$ (cf. (18)) we get from (20) that

$$H_{xx} + H_{yy} = 0. \quad (22)$$

Now we shall show that

$$H_{xx} = H_{xy} = H_{yy} = 0.$$

Indeed, differentiating (21) in x and y gives, in view of (22), that

$$H_x H_{xx} + H_y H_{xy} = 0, \qquad -H_y H_{xx} + H_x H_{xy} = 0.$$

Therefore

$$(H_x^2 + H_y^2)H_{xx} = (H_x^2 + H_y^2)H_{xy} = 0. \quad (23)$$

Suppose $H_{xx}(x_0, y_0, z_0, t_0) \neq 0$ at some point (x_0, y_0, z_0, t_0). Then $H_{xx} \neq 0$ on a neighborhood $\mathcal{U}$ of this point and (23) implies that $H_x = 0$ on $\mathcal{U}$, and hence $H_{xx}(x_0, y_0, z_0, t_0) = 0$, a contradiction. Thus $H_{xx} = H_{xy} = H_{yy} = 0$; therefore H is a linear function in x and y, say

$$H = xp(z,t) + yq(z,t) + 2\xi(z,t).$$

Then it follows from (20) that

$$a = \frac{1}{4}(x^2 - y^2)\alpha + \frac{1}{2}x(p + \delta) + \frac{1}{2}y(q - \beta) + \xi \quad (24)$$

and (21) gives

$$p^2 + q^2 = 2\delta_z + 2\beta_t - \alpha T. \quad (25)$$

Now, integrating the system (18) for c, we obtain that

$$c = \frac{1}{2}xy\alpha + \frac{1}{2}x(q+\beta) - \frac{1}{2}y(p-\delta) + \gamma, \tag{26}$$

where γ is a smooth function depending only on z and t. Setting

$$4\kappa = \alpha, \qquad 2P = p+\delta, \qquad 2Q = q-\beta, \qquad 2U = q+\beta, \qquad 2V = -p+\delta, \qquad \eta = T-\xi$$

we see from (24) to (26) that the functions a, b and c have the forms (14) and (15).

Let τ and τ^* be the scalar and $*$-scalar curvatures. Then, it follows from (13) and (14) that $\tau = \tau^* = 8\kappa = \text{const}$. □

Corollary 4. *A proper almost Hermitian structure (g, J) is almost Kähler, self-dual and $*$-Einstein if and only if the functions a, b and c have the forms (14) and (15) with $\kappa = 0$.*

Corollary 5. *If the function a (resp. b) depends only on (z, t), then the structure (g, J) is almost Kähler and $*$-Einstein if and only if the function b (resp. a) depends only on (z, t) and the function c has the form*

$$c = xU(z,t) + yV(z,t) + \gamma(z,t),$$

where

$$2(V_z + U_t) = V^2 + U^2.$$

Corollary 6. *If the function c depends only on (z, t), then the structure (g, J) is almost Kähler and $*$-Einstein if and only if the functions a and b have the forms*

$$\begin{aligned} a &= xP(z,t) + yQ(z,t) + \xi(z,t), \\ b &= -xP(z,t) - yQ(z,t) + \eta(z,t), \end{aligned} \tag{27}$$

where

$$2(P_z - Q_t) = P^2 + Q^2. \tag{28}$$

6. Almost Kähler Einstein proper structures

The Einstein equation for a Walker metric (1) is a system of PDEs as follows (cf. [22]):

$$\begin{aligned}
(\rho_0)_{xz} &= -(\rho_0)_{yt} = (\rho_0)_{zx} = -(\rho_0)_{ty} = \frac{1}{4}(a_{xx} - b_{yy}) = 0, \\
(\rho_0)_{xt} &= (\rho_0)_{tx} = \frac{1}{2}(b_{xy} + c_{xx}) = 0, \\
(\rho_0)_{yz} &= (\rho_0)_{zy} = \frac{1}{2}(a_{xy} + c_{yy}) = 0, \\
(\rho_0)_{zz} &= \frac{1}{4}\Big\{2a_xc_y + 2a_yb_y - 2a_yc_x - 2c_y^2 + aa_{xx} + 4ca_{xy} \\
&\quad + 2ba_{yy} - 4a_{yt} - a\,b_{yy} - 2ac_{xy} + 4c_{yz}\Big\} = 0, \\
(\rho_0)_{zt} &= (\rho_0)_{tz} = \frac{1}{4}\{-2a_yb_x + 2c_xc_y - ca_{xx} + 2a_{xt} - cb_{yy} \\
&\quad + 2b_{yz} + 2ac_{xx} + 2cc_{xy} - 2c_{xz} + 2bc_{yy} - 2c_{yt}\} = 0, \\
(\rho_0)_{tt} &= \frac{1}{4}\Big\{2a_xb_x - 2b_xc_y + 2b_yc_x - 2c_x^2 - ba_{xx} + 2ab_{xx} \\
&\quad + 4cb_{xy} - 4b_{xz} + bb_{yy} - 2bc_{xy} + 4c_{xt}\Big\} = 0.
\end{aligned} \tag{29}$$

Note that the scalar curvature is given by

$$\tau = a_{xx} + b_{yy} + 2\,c_{xy}. \tag{30}$$

Theorem 7. *A proper structure (g, J) is strictly almost Kähler Einstein if and only if the functions a, b and c have the forms*

$$\begin{aligned} a &= xP(z,t) + yQ(z,t) + \xi(z,t),\\ b &= -xP(z,t) - yQ(z,t) + \eta(z,t),\\ c &= xU(z,t) + yV(z,t) + \gamma(z,t), \end{aligned} \tag{31}$$

where

$$\begin{aligned} 2(V_z - Q_t) &= V^2 - VP + Q^2 + UQ,\\ 2(P_z + U_t) &= P^2 - VP + U^2 + UQ,\\ Q_z + U_z - P_t + V_t &= PQ + UV, \end{aligned} \tag{32}$$

and $(V - P)^2 + (U + Q)^2 \not\equiv 0$.

Proof. It follows from (4) and (29) that the structure (g, J) is almost Kähler Einstein if and only if

$$a_x + b_x = a_y + b_y = 0, \qquad a_{xy} + c_{yy} = b_{xy} + c_{xx} = 0, \qquad a_{xx} = b_{yy} \tag{33}$$

and

$$\begin{aligned} &ba_{yy} + 2ca_{xy} - ac_{xy} - 2a_{yt} + 2c_{yz} + a_y b_y + a_x c_y - a_y c_x - c_y^2 = 0,\\ &ab_{xy} + ba_{xy} + ca_{xx} - cc_{xy} - a_{xt} - b_{yz} + c_{yt} + c_{xz} + a_y b_x - c_x c_y = 0,\\ &ab_{xx} + 2cb_{xy} - bc_{xy} - 2b_{xz} + 2c_{xt} + a_x b_x - b_x c_y + c_x b_y - c_x^2 = 0. \end{aligned} \tag{34}$$

It is easy to see that Eq. (33) implies (16). Moreover adding up the first and the third equations of (34) we get (17). Hence the structure (g, J) is $*$-Einstein and the functions a, b, c have the forms (14). Substituting the expressions (14) for a, b, c into the first equation of (34) and comparing the coefficients of the variables x and y, we get $\kappa(Q + U) = \kappa(P - V) = 0$. It follows that $\kappa = 0$ since otherwise $Q + U = P - V = 0$ which implies, by (6), that the structure (g, J) is Kähler. Thus the functions a, b, c have the forms (31). Now it is easy to check that the system (34) takes the form (32). □

Corollary 8. *Any proper strictly almost Kähler Einstein structure is self-dual, Ricci flat and $*$-Ricci flat.*

Corollary 9. *If the function c depends only on (z, t), the structure (g, J) is strictly almost Kähler Einstein if and only if the functions a and b have the forms*

$$\begin{aligned} a &= xP(z,t) + yQ(z,t) + \xi(z,t)\\ b &= -xP(z,t) - yQ(z,t) + \eta(z,t), \end{aligned} \tag{35}$$

where

$$2P_z = P^2, \qquad 2Q_t = -Q^2, \qquad Q_z - P_t = PQ, \tag{36}$$

and $P^2 + Q^2 \not\equiv 0$.

Remark 2. In a neighborhood of a point where $P^2 + Q^2 \neq 0$ the solution of the system (36) is given by

$$P = -\frac{2p}{pz + qt + r}, \qquad Q = \frac{2q}{pz + qt + r}, \tag{37}$$

where p, q, r are constants and $p^2 + q^2 \neq 0$.

Indeed, suppose, for example, that $P(z_0, t_0) \neq 0$. Then the first equation of (36) implies that, in a neighborhood of (z_0, t_0), $P(z, t) = -2(z + \xi(t))^{-1}$, where $\xi(t)$ is a smooth function. Then the third equation of (36) can be written as $[Q(z, t)(z + \xi(t))^2]_z = 2\xi'(t)$, and hence $Q(z, t) = [2\xi'(t)z + \eta(t)][z + \xi(t)]^{-2}$, where $\eta(t)$ is a smooth function. Now putting the expressions for P and Q into the second equation of (36) and comparing the coefficients of z^3 on the either side, we get $\xi'' = 0$. Therefore $\xi = \alpha t + \beta$ where $\alpha =$ const, $\beta =$ const. Then the second equation of (36) implies that $\eta'(t) = 2\alpha^2$, and thus $\eta(t) = 2\alpha^2 t + \delta$, $\delta =$ const. Now using again this equation, we obtain $\delta = 2\alpha\beta$. Therefore $P = -2(z + \alpha t + \beta)^{-1}$ and $Q = 2\alpha(z + \alpha t + \beta)^{-1}$. Putting $\alpha = \frac{q}{p}$ we see that the functions P and Q have the forms (37).

Note that setting $q = r = 0$ in (37) and $\xi = \eta = 0$ in (36), we obtain the first example of a Ricci flat, $*$-Ricci flat, and strictly almost Kähler indefinite structure due to Haze (see [22]).

Corollary 10. *If the function a (resp. b) depends only on (z, t), then the structure (g, J) is strictly almost Kähler Einstein if and only if the function b (resp. a) depends only on (z, t) and the function c has the form*

$$c = xU(z, t) + yV(z, t) + \gamma(z, t), \tag{38}$$

where

$$2U_t = U^2, \qquad 2V_z = V^2, \qquad U_z + V_t = UV, \tag{39}$$

and $U^2 + V^2 \not\equiv 0$.

Remark 3. In a neighborhood of a point where $U^2 + V^2 \neq 0$ the solution of the system (39) is given by

$$U = -\frac{2p}{qz + pt + r}, \qquad V = -\frac{2q}{qz + pt + r}, \tag{40}$$

where p, q, r are constants and $p^2 + q^2 \neq 0$.

7. Almost Kähler proper structures of constant curvature

It is well known [2,24,25] that in the definite case there are no strictly almost Kähler structures of constant Riemannian sectional curvature. In the indefinite case, however, one can construct such structures as we shall show in this section. For this purpose we consider special classes of proper almost Hermitian structures (g, J) defined by (1) and (3).

Theorem 11. *If $c = 0$, the structure (g, J) is strictly almost Kähler and of constant Riemannian sectional curvature if and only if the functions a and b have the forms*

$$\begin{aligned} a &= xP(z, t) + yQ(z, t) + \xi(z, t) \\ b &= -xP(z, t) - yQ(z, t) + \eta(z, t), \end{aligned} \tag{41}$$

where P, Q, ξ, η satisfy the following PDEs

$$\begin{aligned} &2P_z = P^2, \qquad 2P_t = -PQ, \qquad 2Q_z = PQ, \qquad 2Q_t = -Q^2 \\ &2\xi_{tt} + 2\eta_{zz} + (\xi_z + \eta_z)P - (\xi_t + \eta_t)Q + \xi P^2 + \eta Q^2 = 0 \end{aligned} \tag{42}$$

and $P^2 + Q^2 \not\equiv 0$.

In particular, every such a structure (g, J) is flat.

Proof. Suppose the structure (g, J) is strictly almost Kähler and of constant Riemannian sectional curvature. Then it is Einstein and according to Corollary 9 the functions a and b have the forms (35), where P and Q satisfy the Eqs. (36). Since $W^+ = 0$, substituting a and b into (10) and comparing the coefficients of x and y, we get

$$\begin{aligned} &PP_z + PQ_t + P_{tt} - P_{zz} = 0, \qquad QP_z + QQ_t + Q_{tt} - Q_{zz} = 0, \qquad P_t + Q_z = 0, \\ &2\xi_{tt} + 2\eta_{zz} + P(\xi_z + \eta_z) - Q(\xi_t + \eta_t) + 2\xi P_z - 2\eta Q_t = 0. \end{aligned} \tag{43}$$

Taking into account the Eqs. (36) one can easily check that the Eqs. (43) are equivalent to (42). The converse statement follows from Corollaries 8 and 9, and formulas (10). □

Using similar arguments one can prove the following

Theorem 12. *If $a = b = 0$, the structure (g, J) is of constant Riemannian sectional curvature if and only if the function c has the form*

$$c = xU(z,t) + yV(z,t) + \gamma(z,t),$$

where U, V, and γ satisfy the following PDEs:

$$2U_z = UV, \qquad 2U_t = U^2, \qquad 2V_z = V^2, \qquad 2V_t = UV, \qquad \gamma_{zt} = \gamma U_z$$

and $U^2 + V^2 \not\equiv 0$.

In particular, every such a structure (g, J) is flat.

Example. Let p, q, r be arbitrary constants and $p^2 + q^2 \neq 0$. Set

$$a = \frac{-2px + 2qy}{pz + qt + r}, \qquad b = \frac{2px - 2qy}{pz + qt + r}, \qquad c = 0,$$

or

$$a = 0, \qquad b = 0, \qquad c = -\frac{2px + 2qy}{qz + pt + r}.$$

Then it follows from Theorems 11 and 12, and Remarks 2 and 3 that the proper almost Hermitian structure (g, J) determined by means of the functions a, b, c defined above is strictly almost Kähler and the metric g is flat.

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